

On traceability of claw- o_{-1} -heavy graphs ^{*}

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Abstract

A graph is called traceable if it contains a Hamilton path, i.e., a path passing through all its vertices. Let G be a graph on n vertices. G is called claw- o_{-1} -heavy if every induced claw ($K_{1,3}$) of G has a pair of nonadjacent vertices with degree sum at least $n - 1$ in G . In this paper we show that a claw- o_{-1} -heavy graph G is traceable if we impose certain additional conditions on G involving forbidden induced subgraphs.

Keywords: Traceable graphs; Claw- o_{-1} -heavy graphs; Forbidden subgraphs

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

A graph G is *traceable* if it contains a *Hamilton path*, i.e., a path containing all vertices of G ; and it is *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all vertices of G .

Here we first shortly describe some types of sufficient conditions for the existence of Hamilton cycles, one of which have been popular research areas for a considerable time, namely *forbidden subgraph conditions*. Before we do so, we need to introduce some additional terminology.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph of G induced by $V(G')$). For a given graph H , we say that G is *H -free* if G does not contain an induced subgraph isomorphic to H . For a family \mathcal{H} of graphs, G is called *\mathcal{H} -free* if G is H -free for every $H \in \mathcal{H}$. If G is H -free, then H is called a *forbidden subgraph* of G .

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The graph $K_{1,3}$ is called a *claw*, in which the only vertex of degree 3 is called the *center* and the other vertices are the *end vertices*.

Forbidden subgraph conditions for hamiltonicity have been known since the early 1980s, but Bedrossian was the first to study the characterization of all pairs of forbidden graphs for hamiltonicity in his PhD thesis of 1991 [1]. Before we state his result, we first note that forbidding K_1 is absurd because we always assume a graph has a nonempty vertex set. Moreover, we note that a K_2 -free graph is an empty graph (contains no edges), so it is trivially non-hamiltonian. In this paper, we therefore assume that all the forbidden subgraphs we will consider have at least three vertices. Finally, we note that every connected P_3 -free graph is complete, and then is trivially hamiltonian (if it has at least 3 vertices), and it is in fact easy to show that P_3 is the only connected graph S such that every 2-connected S -free graph is hamiltonian. The next result of Bedrossian deals with pairs of forbidden subgraphs, excluding P_3 .

Theorem 1 (Bedrossian [1]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$ -free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W .*

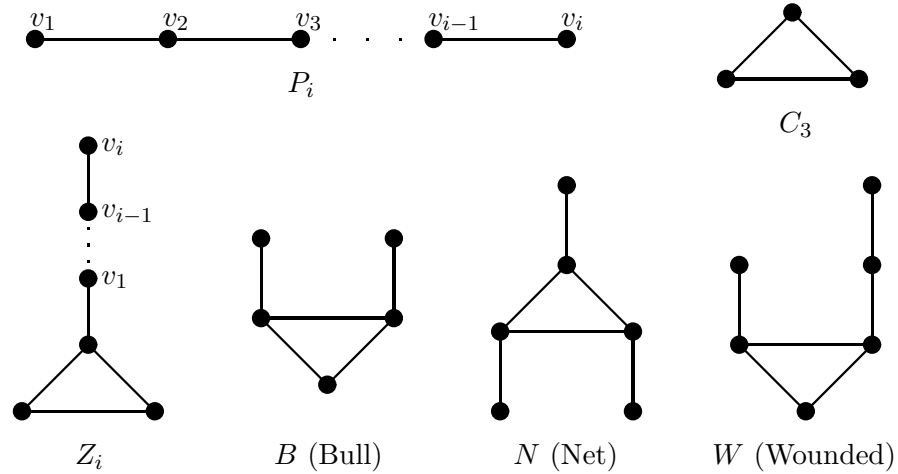


Fig. 1. Graphs P_i, C_3, Z_i, B, N and W

A well-known sufficient condition for a graph to be hamiltonian was given by Ore [6] in 1960, and was called *degree sum condition*. It states that a graph G on $n \geq 3$ vertices is hamiltonian if every pair of nonadjacent vertices of G has degree sum at least n .

In an earlier paper [5], we combine the two types of conditions, i.e., to restrict the degree sum condition to certain subgraphs, to obtain a new type of conditions for hamiltonicity that we generally address as *heavy subgraph conditions*. Before we present the results of it, we need a few more definitions.

Let G be a graph on n vertices, and let G' be an induced subgraph of G . We say that G' is *heavy* in G if there are two nonadjacent vertices in $V(G')$ with degree sum at least n in G . For a given fixed graph H , the graph G is called *H -heavy* if every induced subgraph of G isomorphic to H is heavy. For a family \mathcal{H} of graphs, G is called *\mathcal{H} -heavy* if G is H -heavy for every $H \in \mathcal{H}$. Note that an H -free graph is also H -heavy; and if H_1 is an induced subgraph of H_2 , then an H_1 -free (H_1 -heavy) graph is also H_2 -free (H_2 -heavy).

For hamiltonicity we obtained the following counterpart of Bedrossian's Theorem (it was also shown in [5] that the only connected graph S such that every 2-connected S -free graph is hamiltonian is P_3).

Theorem 2 (Li et al. [5]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$ -heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or W .*

Comparing the two theorems, we note that the claw $K_{1,3}$ is always one of the heavy pairs, and P_6 is the only graph that appears in the list of Bedrossian's Theorem but is missing here. One can find an example in [5] showing that P_6 has to be excluded in the above theorem.

Now we consider the subgraph conditions for traceability of graphs. First, as pointed out before, if a graph is connected and P_3 -free, then it is a complete graph and of course is traceable. In fact, P_3 is the only connected graph S such that every connected S -free graphs is traceable. The following theorem on forbidden pair of subgraphs for traceability is well known.

Theorem 3 (Duffus, Jacobson and Gould [3]). *If G is a connected $\{K_{1,3}, N\}$ -free graph, then G is traceable.*

Obviously if H is an induced subgraph of N , then $\{K_{1,3}, H\}$ will also solve this problem. Faudree et al. proved these are the only forbidden pairs with such property.

Theorem 4 (Faudree and Gould [4]). *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$ -free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, Z_1, B$ or N .*

A counterpart of Ore's Theorem shows that every graph on n vertices in which every pair of nonadjacent vertices has degree sum at least $n - 1$, is traceable. The main object of this paper, is to restrict the degree sum condition to certain subgraphs, to obtain a new type of conditions for traceability. We first give some definitions.

Let G be a graph on n vertices and G' an induced subgraph of G . We say that G' is *o_{-1} -heavy* if there are two nonadjacent vertices in $V(G')$ with degree sum at least $n - 1$

in G . For a given graph H , G is called H - o_{-1} -heavy if every induced subgraph of G isomorphic to H is o_{-1} -heavy. For a family \mathcal{H} of graphs, G is called \mathcal{H} - o_{-1} -heavy if G is H - o_{-1} -heavy for every $H \in \mathcal{H}$.

In this paper, instead of $K_{1,3}$ -free ($K_{1,3}$ -heavy, $K_{1,3}$ - o_{-1} -heavy), we use the terminology claw-free (claw-heavy, claw- o_{-1} -heavy).

Now we consider the following question: for which graph S (which pair of graphs R, S), a connected graph is S - o_{-1} -heavy ($\{R, S\}$ - o_{-1} -heavy) implies it is traceable?

First, we will prove in Section 4 that every connected P_3 - o_{-1} -heavy graph is traceable.

Theorem 5. *If G is a connected P_3 - o_{-1} -heavy graph, then G is traceable.*

It is not difficult to see that P_3 is the only connected graph S such that every connected S - o_{-1} -heavy graph is traceable. It is more interesting to consider which pair of graphs R and S other than P_3 imply that every connected $\{R, S\}$ - o_{-1} -heavy graph is traceable. In fact, as we show bellow, there is only one such pair of subgraphs.

Theorem 6. *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$ - o_{-1} -heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

Since C_3 is a clique, a graph is C_3 - o_{-1} -heavy is equivalent to it is C_3 -free. Thus for the sufficiency of Theorem 6, we only need to prove that every connected claw- o_{-1} -heavy and C_3 -free graph is traceable. In fact, we can prove a stronger theorem as bellow.

Theorem 7. *If G is a connected claw- o_{-1} -heavy and Z_1 -free graph, then G is traceable.*

We postpone the proof of Theorem 7 in Section 5, and in Section 6, we will prove the following theorem, which shows another subgraph S such that a connected claw- o_{-1} -heavy and S -free graph is traceable.

Theorem 8. *If G is a connected claw- o_{-1} -heavy and P_4 -free graph, then G is traceable.*

In fact, these are the only forbidden subgraphs satisfying such property.

Theorem 9. *Let S be connected graphs with $S \neq P_3$ and let G be a connected claw- o_{-1} -heavy graph. Then G being S -free implies G is traceable if and only if $S = C_3, Z_1$ or P_4 .*

We prove the necessity of Theorems 4 and 7 in Section 3.

2 Some preliminaries

We first give some additional terminology and notation.

Let G be a graph, P be a path of G and $x, y \in V(P)$. We use $P[x, y]$ to denote the subpath of P from x to y .

Let G be a graph on n vertices and k be an integer. We call a sequence of vertices $P = v_1 v_2 \cdots v_k$ an o_{-1} -path of G , if for all $i \in [1, k-1]$, either $v_i v_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n-1$. The *deficit* of P is defined by $\text{def}(P) = |\{i \in [1, k-1] : v_i v_{i+1} \notin E(G)\}|$. Thus a path is an o_{-1} -path with deficit 0.

Now, we prove the following lemma on o_{-1} -paths.

Lemma 1. *Let G be a graph and P an o_{-1} -path of G . Then there exists a path of G which contains all the vertices in $V(P)$.*

Proof. Assume the opposite. Let P' be an o_{-1} -path which contains all the vertices in $V(P)$ such that $\text{def}(P')$ is as small as possible. Then we have $\text{def}(P') \geq 1$. Without loss of generality, we assume that $P' = v_1 v_2 \cdots v_p$ such that $v_k v_{k+1} \notin E(G)$ and $d(v_k) + d(v_{k+1}) \geq n-1$, where $1 \leq k \leq p-1$.

If v_k and v_{k+1} have a common neighbor in $V(G) \setminus V(P)$, denote it by x . Then $P'' = P'[v_1, v_k] v_k x v_{k+1} P'[v_{k+1}, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P)$ with deficit smaller than $\text{def}(P')$, a contradiction.

So we assume that $N_{G-P'}(v_1) \cap N_{G-P'}(v_k) = \emptyset$. Then we have $d_{P'}(v_k) + d_{P'}(v_{k+1}) \geq |V(P')| - 1$ by $d(v_k) + d(v_{k+1}) \geq n-1$.

If $v_1 v_{k+1} \in E(G)$, then $P'' = P'[v_k, v_1] v_1 v_{k+1} P'[v_{k+1}, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P)$ with deficit smaller than $\text{def}(P')$, a contradiction. Thus we assume that $v_1 v_{k+1} \notin E(G)$, and similarly, $v_p v_k \notin E(G)$. Thus, there exists $i \in [1, p-1] \setminus \{k\}$ such that $v_i \in N_P(v_k)$ and $v_{i+1} \in N_P(v_{k+1})$.

If $1 \leq i \leq k-1$, then $P'' = P'[v_1, v_i] v_i v_k P'[v_k, v_{i+1}] v_{i+1} v_{k+1} P'[v_{k+1}, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P)$ with deficit smaller than $\text{def}(P')$, a contradiction. If $k+1 \leq i \leq p-1$, then $P'' = P'[v_1, v_k] v_k v_i P'[v_i, v_{k+1}] v_{k+1} v_{i+1} P'[v_{i+1}, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P)$ with deficit smaller than $\text{def}(P')$, a contradiction. \square

In the following, we use $\tilde{E}_{-1}(G)$ to denote the set $\{uv : uv \in E(G) \text{ or } d(u) + d(v) \geq n-1\}$.

We now give a lemma on claw- o_{-1} -heavy graphs.

Lemma 2. *Let G be a connected claw- o_{-1} -heavy graphs and x be a cut-vertex of G . Then (1) $G - x$ contains exactly two components; and*

(2) if x_1 and x_2 are two neighbors of x in a common component of $G - x$, then $x_1x_2 \in \tilde{E}_{-1}(G)$.

Proof. If there are at least three components of $G - x$, then let H_1, H_2 and H_3 be three components. Let x_1, x_2 and x_3 be neighbors of x in H_1, H_2 and H_3 , respectively. Then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is a claw. Besides, for $1 \leq i < j \leq 3$, $d(x_i) + d(x_j) \leq |V(H_i)| + |V(H_j)| \leq n - 2$, a contradiction. Thus, $G - x$ has exactly two components.

Let x_1, x_2 be two neighbors of x in a common component H . If $x_1x_2 \notin E(G)$, then let x' be a neighbor of x in the other component H' and the subgraph induced by $\{x, x_1, x_2, x'\}$ is a claw. Besides, for $i = 1, 2$, $d(x_i) + d(x') \leq |V(H)| - 1 + |V(H')| \leq n - 2$. Since G is claw- o_{-1} -heavy, we have $d(x_1) + d(x_2) \geq n - 1$. \square

3 The proof of the necessity of Theorems 6 and 9

We construct two non-traceable graphs as follows.

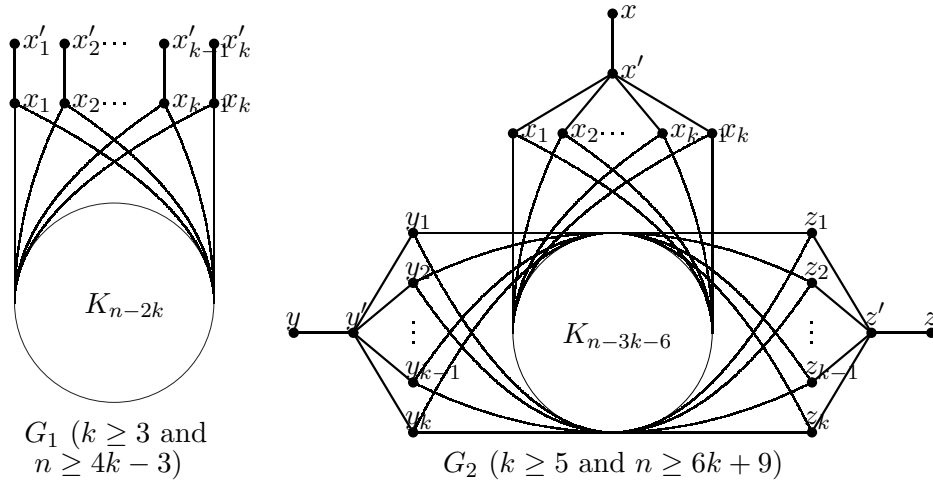


Fig. 2. Two non-traceable graphs.

Let R and S be two connected graphs other than P_3 such that every connected $\{R, S\}$ - o_{-1} -heavy graph is traceable. Then by Theorem 2, up to symmetry, $R = K_{1,3}$ and S be C_3, P_4, Z_1, B or N . Note that G_1 is $\{K_{1,3}, P_4\}$ - o_{-1} -heavy and G_2 is $\{K_{1,3}, Z_1\}$ - o_{-1} -heavy. Thus S must be C_3 .

Let S be a connected graph other than P_3 such that every connected claw- o_{-1} -heavy and S -free graph is traceable. By Theorem 2, S must be C_3, P_4, Z_1, B or N . Note that G_1 is B -free. Thus S must be C_3, P_4 or Z_1 .

4 Proof of Theorem 5

We use n to denote the order of G . Let $P = v_1v_2 \cdots v_p$ be a longest path of G . Assume that G is not traceable. Then $V(G) \setminus V(P) \neq \emptyset$. Since G is connected, there exists a vertex $x \in V(G - P)$ joined to P . Let v_i be a neighbor of x in P . Clearly $v_i \neq v_p$, otherwise $P' = Pv_px$ is a path longer than P . If $xv_{i+1} \in E(G)$, then $P' = P[v_1, v_i]v_ixv_{i+1}P[v_{i+1}, v_p]$ is a path longer than P , a contradiction. Thus we assume that $xv_{i+1} \notin E(G)$. Since G is P_3 - o_{-1} -heavy, we have that $d(x) + d(v_{i+1}) \geq n - 1$. Thus $P' = P[v_1, v_i]v_ixv_{i+1}P[v_{i+1}, v_p]$ is an o_{-1} -path of G . By Lemma 2, there is a path of G containing all the vertices in P' , a contradiction.

5 Proof of Theorem 7

If G contains only one or two vertices, then the result is trivially true. So we assume that G contains at least three vertices. We use n to denote the order of G . We distinguish two cases.

Case 1. G is separable.

If G itself is a path, then we have nothing need to prove. Thus we assume that G is not a path. Thus there must be a cut-vertex of G with degree at least 3. Let x be such a cut-vertex. By Lemma 1, $G - x$ has exactly two components. Let C and D be the two components of $G - x$. Since $d(x) \geq 3$, without loss of generality, we assume that x has at least two neighbors in D .

If x contained in a triangle $xx'x''$, then x' and x'' is in a common component of $G - x$. Without loss of generality, Let $x', x'' \in V(D)$. Let w be a neighbor of x in C . Then the subgraph induced by $\{x, x', x'', w\}$ is a Z_1 , a contradiction. Thus we assume that x is not contained in a triangle and $N(x)$ is an independent set.

Let y be a neighbor of x . If y contained in a triangle $yy'y''$, then clearly $xy', xy'' \notin E(G)$, otherwise x will be contained in a triangle. Thus the subgraph induced by $\{y, y', y'', x\}$ is a Z_1 , a contradiction. Thus we assume that y is not contained in a triangle and $N(y)$ is an independent set. Similarly, let z be a vertex with distance 2 from x , and y be a common neighbor of x and z . If z is contained in a triangle $zz'z''$, then clearly $yz', yz'' \notin E(G)$, otherwise y will be contained in a triangle. Thus the subgraph induced by $\{z, z', z'', y\}$ is a Z_1 , a contradiction. Thus we assume that z is not contained in a triangle and $N(z)$ is an independent set. Thus we have that every vertex adjacent to x or with distance 2 from x is not contained in a triangle.

Let w be a neighbor of x in C and y be a neighbor of x in D . Let y' be a neighbor of x in D other than y . Since $yy' \notin E(G)$, by Lemma 1, we have that $d(y) + d(y') \geq n - 1$. Without loss of generality, we assume that $d(y) \geq (n - 1)/2$. Note that x and y have no common neighbors, we have $d(x) \leq (n + 1)/2$.

Case A. $d(x) = (n + 1)/2$.

In this case, n is odd. Let $Y = N(x) \setminus \{w\}$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Then $|Y| = (n - 1)/2$ and $|Z| = (n - 3)/2$. Since $d(y) \geq (n - 1)/2$ and y is nonadjacent to any vertex in $Y \cup \{w\}$, we have that y is adjacent to every vertex in Z and $d(y) = (n - 1)/2$. This implies that $Z \subset V(D)$. Thus every vertex in $N_C(x)$ will have degree 1. This implies that w is the only vertex in C and $Y \subset V(D)$.

Note that $d(y) = (n - 1)/2$. Let y' be a vertex in Y other than y . By Lemma 1, $d(y) + d(y') \geq n - 1$. Thus $d(y') \geq (n - 1)/2$. Since y' is nonadjacent to any vertices in $Y \cup \{w\}$, y' is adjacent to every vertex in Z . This implies that every vertex in Y and every vertex in Z are adjacent.

Let $Y = \{y_1, y_2, \dots, y_{(n-1)/2}\}$ and $Z = \{z_1, z_2, \dots, z_{(n-3)/2}\}$. Then $P = wxy_1z_1y_2z_2 \cdots z_{(n-3)/2}y_{(n-1)/2}$ is a Hamilton path of G .

Case B. $d(x) = n/2$.

In this case, n is even and $d(y) \geq n/2$. Let $Y = N(x) \setminus \{w\}$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Then $|Y| = (n - 2)/2$ and $|Z| = (n - 2)/2$. Since $d(y) \geq n/2$ and y is nonadjacent to any vertices in $Y \cup \{w\}$, we have that y is adjacent to every vertices in Z and $d(y) = n/2$. This implies that $Z \subset V(D)$. Thus every vertex in $N_C(x)$ will have degree 1 and then w is the only vertex in C and $Y \subset V(D)$. Note that $d(x) \geq 3$, $n \geq 6$ and $|Z| \geq 2$.

Let $Y = \{y_1, y_2, \dots, y_{(n-2)/2}\}$, where y_1 is with the minimum degree and $Z = \{z_1, z_2, \dots, z_{(n-2)/2}\}$ where z_1 is with the maximum degree. For every vertex y_i in Y other than y_1 , since $d(y_1) + d(y_i) \geq n - 1$, we have $d(y_i) \geq n/2$. Since y_i is nonadjacent to any vertices in $Y \cup \{w\}$, y_i is adjacent to every vertex of Z . This implies that every vertex in $Y \setminus \{y_1\}$ and every vertex in Z are adjacent to each other.

Let z_i be a vertex of Z other than z_1 . Then the subgraphs induced by $\{y, x, z_1, z_i\}$ is a claw. Since $d(x) = n/2$, we have that $d(z_1) \geq (n - 2)/2$. Note that z_1 is nonadjacent to any vertex in $Z \cup \{x, w\}$, we have that z_1 is adjacent to every vertex in Y and then $y_1z_1 \in E(G)$.

Thus $P = wxy_1z_1y_2z_2 \cdots y_{(n-2)/2}z_{(n-2)/2}$ is a Hamilton path of G .

Case C. $d(x) \leq (n - 1)/2$.

Note that $d(x) \geq 3$, we have $n \geq 7$ and $d(y) \geq (n - 1)/2 \geq 3$. Let z be a neighbor of y other than x with the maximum degree. Let z' be a neighbor of y other than x and z . Then the subgraph induced by $\{y, x, z, z'\}$ is a claw. Since $d(x) \leq (n - 1)/2$, we have that

$d(z) \geq (n-1)/2$.

Let $Y = N(z)$ and $Z = V(G) \setminus Y \setminus \{x, w\}$. Note that $d(y) \geq (n-1)/2$ and y is nonadjacent to any vertices in $Y \cup \{w\}$ and $d(z) \geq (n-1)/2$ and z is nonadjacent to any vertices in $Z \cup \{x, w\}$. We have that $|Y| = (n-1)/2$, $|Z| = (n-3)/2$ and y is adjacent to every vertex in Z . This implies that there is only the one vertex w in C and $Y, Z \subset V(D)$.

Note that x has at least two neighbors in Y . Let $Y = \{y_1, y_2, \dots, y_{(n-1)/2}\}$, where y_1 and y_2 are two neighbors of x and $Z = \{z_1, z_2, \dots, z_{(n-3)/2}\}$ where z_1 is with the minimum degree. Since $d(y_1) + d(y_2) \geq n-1$ and y_1 and y_2 are nonadjacent to any vertices in $Y \cup \{w\}$, we have that y_1 and y_2 are adjacent to any vertices in Z , and then $y_1 z_1, y_2 z_1 \in E(G)$.

Let z_i be a vertex of Z other than z_1 . Then the subgraphs induced by $\{y, x, z_1, z_i\}$ is a claw. Since $d(x) \leq (n-1)/2$, we have that $d(z_i) \geq (n-1)/2$. Note that z_i is nonadjacent to any vertices in $Z \cup \{x, w\}$, we have that z_i is adjacent to every vertex in Y . This implies that every vertices in Y and every vertex in $Z \setminus \{z_1\}$ are adjacent.

Thus $P = wxy_1z_1y_2z_2 \cdots z_{(n-3)/2}y_{(n-1)/2}$ is a Hamilton path of G .

Case 2. G is 2-connected.

Let $P = v_1v_2 \cdots v_p$ be a longest path of G . Assume that G is not traceable. Then $V(G) \setminus V(P) \neq \emptyset$. Since G is 2-connected, there exists a path R with two end-vertices in P and of length at least 2 which is internally disjoint with P . Let $R = x_0x_1x_2 \cdots x_{r+1}$, where $x_0 = v_i$ and $x_{r+1} = v_j$, be such a path as short as possible. Clearly $i \neq 1, p$ and $j \neq 1, p$. Without loss of generality, we assume that $2 \leq i < j \leq p-1$.

Claim 1. Let $x \in V(R) \setminus \{v_i, v_j\}$ and $y \in \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$. Then $xy \notin \tilde{E}_{-1}(G)$.

Proof. Without loss of generality, we assume $y = v_{i-1}$. If $xv_{i-1} \in \tilde{E}_{-1}(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}xR[x, v_i]v_iP[v_i, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P) \cup V(R[x, v_i])$. By Lemma 2, there is a path containing all the vertices in P' , a contradiction. \square

Claim 2. $v_{i-1}v_{i+1} \in \tilde{E}_{-1}(G)$, $v_{j-1}v_{j+1} \in \tilde{E}_{-1}(G)$.

Proof. If $v_{i-1}v_{i+1} \notin E(G)$, by Claim 1, the graph induced by $\{v_i, x_1, v_{i-1}, v_{i+1}\}$ is a claw, where $d(x_1) + d(v_{i\pm 1}) < n-1$. Since G is a claw- o_{-1} -heavy graph, we have that $d(v_{i-1}) + d(v_{i+1}) \geq n$.

The second assertion can be proved similarly. \square

Claim 3. $v_{i-1}v_{j-1} \notin \tilde{E}_{-1}(G)$, $v_{i+1}v_{j+1} \notin \tilde{E}_{-1}(G)$.

Proof. If $v_{i-1}v_{j-1} \in \tilde{E}_{-1}(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{j-1}P[v_{j-1}, v_i]v_iRv_jP[v_j, v_p]$ is an o_{-1} -path which contains all the vertices in $V(P) \cup V(R)$, a contradiction.

The second assertion can be proved similarly. \square

Claim 4. Either $v_{i-1}v_{i+1} \in E(G)$ or $v_{j-1}v_{j+1} \in E(G)$

Proof. Assume the opposite. By Claim 2 we have $d(v_{i-1}) + d(v_{i+1}) \geq n - 1$ and $d(v_{j-1}) + d(v_{j+1}) \geq n - 1$. By Claim 3, we have $d(v_{i-1}) + d(v_{j-1}) < n - 1$ and $d(v_{i+1}) + d(v_{j+1}) < n - 1$, a contradiction. \square

Without loss of generality, we assume that $v_{i-1}v_{i+1} \in E(G)$. Then the subgraph induced by $\{v_i, v_{i-1}, v_{i+1}, x_1\}$ is a Z_1 , a contradiction.

The proof is complete.

6 Proof of Theorem 8

If G contains only one or two vertices, then the result is trivially true. So we assume that G contains at least three vertices. We use n to denote the order of G . We distinguish two cases.

Case 1. G is separable.

Let x be a cut-vertex of G . By Lemma 1, $G - x$ has exactly two components. Let C and D be the two components of $G - x$.

If there is a vertex in D which is nonadjacent to x , then let z be a vertex in D with distance 2 from x , and y be a common neighbor of x and z . Let w be a neighbor of x in C . Then $wxyz$ is an induced P_4 of G , a contradiction. Thus we have that x is adjacent to every vertex in D . By Lemma 1, for every two vertices y and y' in D , $yy' \in \tilde{E}_{-1}(G)$. Similarly, x is adjacent to every vertex in C and for every two vertices w and w' in C , $ww' \in \tilde{E}_{-1}(G)$.

Let $V(C) = \{w_1, w_2, \dots, w_k\}$ and $V(D) = \{y_1, y_2, \dots, y_l\}$, where $k + l + 1 = n$. Then $P' = w_1w_2 \cdots w_kxy_1y_2 \cdots y_l$ is an o_{-1} -path of G . By Lemma 2, there is a path P containing all the vertices in P' , which is a Hamilton path.

Case 2. G is 2-connected.

Let $P = v_1v_2 \cdots v_p$ be a longest path of G . Assume that G is not traceable. Then $V(G) \setminus V(P) \neq \emptyset$. Since G is 2-connected, there exists a path R with two end-vertices in P and of length at least 2 which is internally disjoint with P . Let $R = x_0x_1x_2 \cdots x_{r+1}$,

where $x_0 = v_i$ and $x_{r+1} = v_j$. Clearly $i \neq 1, p$ and $j \neq 1, p$. Without loss of generality, we assume that $2 \leq i < j \leq p-1$.

Similar as in Section 5, we can prove that

Claim 1. Let $x \in V(R) \setminus \{v_i, v_j\}$ and $y \in \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$. Then $xy \notin \tilde{E}_{-1}(G)$.

Claim 2. $v_{i-1}v_{i+1} \in \tilde{E}_{-1}(G)$, $v_{j-1}v_{j+1} \in \tilde{E}_{-1}(G)$.

Now we prove that

Claim 3. $v_iv_{j-1} \notin E(G)$.

Proof. If $v_iv_{j-1} \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{j-1}]v_{j-1}v_jP[v_j, v_p]$ is an o_{-1} -path containing all the vertices in $V(P) \cup V(R)$, a contradiction. \square

Let v_k be the first vertex in $P[v_{i+1}, v_{j-1}]$ which is nonadjacent to v_i . We have that $i+2 \leq k \leq j-1$.

Claim 4. $x_1v_{k-1} \notin E(G)$, $x_1v_k \notin E(G)$.

Proof. If $v_{k-1} = v_{i+1}$, then by Claim 1, we have $x_1v_{i+1} \notin E(G)$. If $i+2 \leq k-1 \leq j-2$ and $x_1v_{k-1} \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{k-2}]v_{k-2}v_ix_1v_{k-1}P[v_{k-1}, v_p]$ is an o_{-1} -path containing all the vertices in $V(P) \cup V(R)$, a contradiction. Thus we have that $x_1v_{k-1} \notin E(G)$.

If $z_1v_k \in E(G)$, then $P' = P[v_1, v_{i-1}]v_{i-1}v_{i+1}P[v_{i+1}, v_{k-1}]v_{k-1}v_ix_1v_kP[v_k, v_p]$ is an o_{-1} -path containing all the vertices in $V(P) \cup V(R)$, a contradiction. Thus we have that $x_1v_k \notin E(G)$. \square

Thus $x_1v_iv_{k-1}v_k$ is an induced P_4 , a contradiction.

The proof is complete.

7 Remark

Here we explain why we use the concept o_{-1} -heavy. In fact one can similarly define o_r -heavy subgraphs for an integer r .

Let G be a graph on n vertices, G' be an induced subgraph of G and r be a given integer. We say that G' is o_r -heavy if there are two nonadjacent vertices in $V(G')$ with degree sum at least $n+r$. For a given graph H , the graph G is called H - o_r -heavy if every induced subgraph of G isomorphic to H is o_r -heavy. For a family \mathcal{H} of graphs, G is called \mathcal{H} - o_r -heavy if G is H - o_r -heavy for every $H \in \mathcal{H}$. Clearly, an H -free graph is H - o_r -heavy for any integer r ; and if $r \leq s$, then an H - o_s -heavy graph is also H - o_r -heavy.

Consider the bipartite graph $K_{k,k+2}$. Note that every subgraph of $K_{k,k+2}$ (other than K_1 and K_2) is o_{-2} -heavy and $K_{k,k+2}$ is non-traceable. Thus, for any class \mathcal{H} of graphs, a connected \mathcal{H} - o_r -heavy graph is not necessarily traceable for $r \leq -2$.

Now we consider the o_r -heavy subgraph conditions for $r \geq 0$. We will show that we cannot get any other pares of subgraphs solving our problem in Theorems 6 and 9.

Theorem 10. *Let $r \geq 0$ be an integer. Let R and S be connected graphs with $R, S \neq P_3$ and let G be a connected graph. Then G being $\{R, S\}$ - o_r -heavy implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3$.*

Theorem 11. *Let $r \geq 0$ be an integer. Let S be a connected graph with $S \neq P_3$ and let G be a connected claw- o_r -heavy graph. Then G being S -free implies G is traceable if and only if $S = C_3, Z_1$ or P_4 .*

The necessity of these two theorems is deduced by Theorems 6 and 9 immediately. Here we show the sufficiency of them. In Fig. 2, take $r \geq 3$ and $n \geq 4k + r - 2$ in G_1 , and take $k \geq r + 6$ and $n \geq 6k + r + 10$ in G_2 . Then G_1 is $\{K_{1,3}, P_4\}$ - o_r -heavy and G_2 is $\{K_{1,3}, Z_1\}$ - o_r -heavy. Since the two graphs are both non-traceable, we get the sufficiency of Theorems 10 and 11.

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